2.3: Interpretations of the Derivative

For a function f, we have seen the derivative f' represented as the instantaneous rate of change. That is to say that $f'(x)$ is the limit as we take the average rate of change in smaller and smaller intervals containing x. For this reason you can think of

$$f'(x) \approx \frac{\Delta y}{\Delta x}.$$

However, there is another notation for derivatives, introduced by the German mathematician Leibniz. When $y = f(x)$ he writes

$$f'(x) = \frac{dy}{dx}.$$

With this notation it is easy to see that the derivative is represented as

$$\frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}.$$

In order to evaluate the Leibniz notation at a point, such as $f'(2) = 4$ say the derivative of f at 2 is 4, we write

$$\left.\frac{dy}{dx}\right| \quad = 4.$$

Using Units to Interpret the Derivative

Example 1: The cost C (in dollars) of building a house A square feet in area is given by the function $C = f(A)$. What are the units and the practical interpretation of the function $f'(A)$?

In general, the units for the derivative function are the same as the units for average rate of change.

1. The units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable. In other words, the units of dA/dB are the units of A divided by the units of B.

2. If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit.

Example 2: The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say $f'(2000) = 100$?

Example 3: If $q = f(p)$ gives the number of thousands of tons of zinc produced when the price is p dollars per ton, then what are the units and the meaning of

$$\left.\frac{dq}{dp}\right| \quad = 0.2?$$

Example 4: The time, L (in hours), that a drug stays in a person's system is a function of the quantity administered, q, in mg, so $L = f(q)$.

(a) Interpret the statement $f(10) = 6$. Give units for the numbers 10 and 6.

(b) Write the derivative of the function $L = f(q)$ in Leibniz notation. If $f'(10) = 0.5$, what are the units of 0.5?

(c) Interpret the statement $f'(10) = 0.5$ in terms of dose and duration.

Example 5: If the velocity of a body at time t seconds is measured in meters/sec, what are the units of the acceleration of the body?

Tangent Line Approximation: Local Linearity

Recall that the derivative tells us how fast the value of a function is changing. So if a function is relatively tame near a point we can use the derivative to estimate values of the function at nearby points.

Example 6: Suppose that $f(t)$ is a function with $f(25) = 3.6$ and $f'(25) = -0.2$. Estimate $f(26)$ and $f(30)$.

Definition: If $y = f(x)$ and Δx is near 0, then $\Delta y \approx f'(x)\Delta x$. For x near a, we have $\Delta y = f(x) - f(a)$, so

$$f(x) \approx f(a) + f'(a)\Delta x$$

is called the **tangent line approximation** of $f(x)$.

Example 7: For a function $f(x)$, we know that $f(20) = 68$ and $f'(20) = -3$. Estimate $f(21)$, $f(19)$ and $f(25)$.

Recall that for relative change of a function we look at the rate of change as a fraction of the original value of the function. We can define relative change analagously with the derivative function.

Definition: The **relative rate of change** of $y = f(t)$ at $t = a$ is defined to be

$$\text{Relative rate of change of } y \text{ at } a \; = \frac{dy/dt}{y} = \frac{f'(a)}{f(a)}.$$

Example 8: Annual world soybean production, $W = f(t)$, in million tons, is a function of t years since the start of 2000.

(a) Interpret the statements $f(8) = 253$ and $f'(8) = 17$ in terms of soybean production.

(b) Calculate the relative rate of change of W at $t = 8$; interpret it in terms of soybean production.

Example 9: Solar photovoltaic (PV) cells are the world's fastest growing energy source. At time t in years since 2005, peak PV energy-generating capacity worldwide was approximately $E = 4.6e^{0.43t}$ gigawatts. Estimate the relative rate of change of PV energy-generating capacity in 2015 using this model and

(a) $\Delta t = 1$ (b) $\Delta t = 0.1$ (c) $\Delta t = 0.01$

Examply 10: In April 2009, the US Bureau of Economic Analysis announced that the US gross domestic product (GDP) was decreasing at an annual rate of 6.1%. The GDP of the US at that time was 13.84 trillion dollars. Calculate the annual rate of change of the US GDP in April 2009.